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A Deeper Look Into an Insurance Risk Model with Two Types of Claims and FGM Copula Dependency

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Abstract

This paper introduces a novel continuous-time risk model that extends the classical framework by incorporating two types of claims and a dependence structure between claim sizes and inter-claim times using a Farlie-Gumbel-Morgenstern (FGM) copula. The methodology begins with the construction of a Lundberg's equation and the determination of its non-negative roots. Subsequently, the integro-differential equation for the ruin probability is derived, from which the Laplace transform of the ruin probability is obtained. For the specific case of exponentially distributed claim sizes, an explicit analytical expression for the ruin probability is derived to examine the effects of dependence parameters and distributional characteristics. A series of numerical experiments with varying FGM copula parameters demonstrate that the ruin probability decreases as the initial surplus increases and is significantly influenced by the strength of the dependence structure. From a practical perspective, distinguishing between claim types allows insurers to identify which category poses the greatest threat to solvency, thereby supporting more targeted underwriting and accurate capital allocation strategies.

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1. INTRODUCTION

Foundational to ruin theory is the classical Cramér-Lundberg risk model in continuous time, introduced by Filip Lundberg in 1903 and further developed by Harald Cramér in the 1930s [1]. The model assumes that insurance claims arrive randomly according to a Poisson process $\{N(t), t \geq 0\}$ with intensity λ, while claim sizes X_i are independent and identically distributed and independent of the interarrival times T_i. Premium income accrues continuously at rate $c(t)$, which is often taken to be constant with $c > 0$. A central implication is that the probability of ruin decreases exponentially with increasing initial surplus u. The insurer's surplus at time t is modeled as $U(t) = u + ct - S(t)$, where the aggregate claims process is $S(t) = \sum_{i=1}^{N(t)} X_i$ [2]. Despite its relative simplicity, the model has significantly influenced actuarial theory and insurance practice. It is widely used for computing ruin probabilities and quantifying bankruptcy risk. Consequently, this utility has motivated continued research and extensions of the framework. Consistent with this framework, Maulida et al. [3] adopts the Cramér-Lundberg risk model with claim sizes follow a mixtures of two exponential distributions and solves the corresponding integro-differential system numerically to assess solvency risk. The findings show that the probability of ruin decreases exponentially as the initial surplus increases, preserving the classical exponential-decay behavior even under mixture-exponential claim sizes.

Since their inception, insurance companies have undergone substantial development. A key evolution is the diversification of product offerings, which has resulted in distinct claim categories across various lines of business. In health insurance, small claims such as routine treatment, prescription drugs, and general practitioner visits occur frequently but at lower cost, whereas large claims such as major surgery or prolonged serious care are rare yet financially substantial. Jiang and Ma [4], Shija and Jacob [5] investigated ruin probabilities for two-claim-type models under varied surplus processes. In this framework, the aggregate claims at time t are expressed as $S(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{j=1}^{N_2(t)} Y_j$. As in the Cramér–Lundberg assumption, several works assume independence between claim sizes and interarrival times, although this assumption often fails to reflect practical realities in operational data. Empirically, Shi et al. [6] documented a positive, albeit weak, dependence between claim frequency and severity in motor insurance. Frees et al. [7] reported a similar weak association for claim frequency and severity in property insurance.

A standard approach to capture dependence between two random variables is to use a copula, which links their marginal distributions to a joint distribution. In the context of estimating ruin probabilities, the Farlie-Gumbel-Morgenstern (FGM) copula is frequently used to model the dependence between claim sizes and inter-claim times. Various studies have embedded this dependence structure into different risk frameworks. For instance, Cossette et al. [8] applied the FGM copula to the classical Cramér-Lundberg model, while Cossette et al. [9] incorporated it into a risk model with dividend strategies. Furthermore, Chadjiconstantinidis and Vrontos [10] analyzed an Erlang(n) risk model, Ragulina [11] investigates a stochastic premium risk model, and Adékambi and Takouda [12] examined a perturbed risk model. Although these works successfully derived explicit formulas for ruin probability under exponentially distributed claim sizes, their frameworks remained confined to a single claim type.

In this research, we investigate ruin probabilities within a Cramér-Lundberg risk model that incorporates two types of claims. A critical distinction in our framework lies in the dependence structure: while the occurrences of type-I and type-II claims are assumed to be mutually independent processes, we explicitly model the dependence within each claim type, specifically between inter-claim times and claim sizes using the FGM copula. This approach allows us to capture the time-size correlation inherent in each line of business while retaining tractable marginal distributions. We define two claim types by categorizing insurance claims into two groups that exhibit distinct risk characteristics. These distinctions may arise from differences in frequency, claim severity, source of occurrence, or operational handling processes. This separation is intended to render the risk model more realistic and accurate. The primary objective of this study is to derive an explicit analytical expression for the ruin probability within a risk model characterizing two types of claims and FGM copula dependence. To achieve this, we first establish the model assumptions, identify the positive roots of the generalized Lundberg's equation, and formulate the corresponding integro-differential equation. Subsequently, we obtain the Laplace transform and apply these results to the case of exponentially distributed claim sizes. Finally, numerical illustrations are provided to demonstrate the impact of dependence structures on solvency risk.

2. METHODS

The approach implemented in this study involved a comprehensive literature review, gathering pertinent information from diverse sources including books and academic journals related to the subject matter. Additionally, numerical simulations were performed using Python. The methodology employed in this study follows a systematic four-step framework. First, we construct the risk model by defining two independent Poisson processes for claim arrivals and employing the FGM copula to model the dependence between claim sizes and inter-claim times. Second, we derive the Lundberg's equation and utilize a modification of Rouche's theorem to determine the existence of its roots in the complex plane. Third, we formulate the integro-differential equation for the ruin probability and apply the Laplace transform to obtain its solution in the transform domain, the Lagrange interpolation formula is then used to facilitate the inverse transformation, yielding an explicit analytical expression. Finally, numerical evaluations are implemented using Python to compute the ruin probabilities based on the derived explicit formulas. These numerical experiments are designed to simulate various dependence scenarios (independent, positive, negative, and hybrid) and analyze the sensitivity of solvency risk to changes in copula parameters and initial surplus levels.

2.1 Poisson Process and Compound Poisson Process

A stochastic process $\{N(t), t \geq 0\}$ is termed a counting process when $N(t)$ represents the number of events up to time t and satisfies the properties $N(t) \geq 0$, $N(t) \in \mathbb{Z}$, and for $s < t$ one has $N(s) < N(t)$, with the increment $N(t) - N(s)$ giving the number of events occurring on $(s, t]$. Within this framework, the process is said to possess independent increments if for any sequence $0 \le t_0 < t_1 < \cdots < t_n$ the random variables $N(t_1) - N(t_0), \dots, N(t_n) - N(t_{n-1})$ are mutually independent, equivalently, counts over pairwise disjoint time intervals are independent.

Definition 1 [13]. *A Poisson process with intensity λ is a counting process $\{N(t), t \ge 0\}$ such that $N(0) = 0$ and the paths are nondecreasing with $N(s) \le N(t)$ for all $s < t$, the small-interval transition probabilities satisfy $\mathbb{P}(N(t+h) = n+1 \mid N(t) = n) = \lambda h + o(h)$, $\mathbb{P}(N(t+h) = n+m \mid N(t) = n) = o(h)$ for $m > 1$, and $\mathbb{P}(N(t+h) = n \mid N(t) = n) = 1 - \lambda h + o(h)$, and the process has independent increments, meaning that counts over disjoint time intervals are independent.*

If the sequence of interarrival times $\{T_n, n = 1, 2, \dots\}$ corresponds to a Poisson counting process, then $\{T_n\}$ consists of independent identically distributed (i.i.d.) exponential random variables with rate λ. In line with this, if $\{N_1(t), t \ge 0\}$ and $\{N_2(t), t \ge 0\}$ are independent Poisson processes with rates λ_1 and λ_2, then $N(t) = N_1(t) + N_2(t)$ is Poisson process with rate $\lambda_1 + \lambda_2$.

Definition 2 [13]. *A stochastic process $\{S(t), t \ge 0\}$ is called a compound Poisson process if it can be written as $S(t) = \sum_{i=1}^{N(t)} X_i$ for $t \ge 0$, where $\{N(t)\}$ is a Poisson process and $\{X_i\}_{i \ge 1}$ are i.i.d. random variables that are independent of $\{N(t)\}$.*

In this study, this stochastic framework serves as the fundamental for modeling the arrival frequency and aggregate magnitude of both claim types within the surplus process.

2.2 Lundberg's Generalized Equation of Cramér-Lundberg Risk Model

In the Cramér-Lundberg risk model, the surplus process is modeled by

$$U(t) = u + c(t) - \sum_{i=1}^{N(t)} X_i, \tag{1}$$

where u is the initial capital and $N(t)$ counts claims up to time t while the aggregate claims are $S(t) = \sum_{i=1}^{N(t)} X_i$. The premium inflow is deterministic with $c(t) = ct$, for $c > 0$. The claim count starts at $N(0) = 0$ and follows a homogeneous Poisson process of rate λ, and the claim sizes $\{X_i\}$ are i.i.d. with finite mean and independent of $\{N(t)\}$, yielding a compound Poisson structure for $S(t)$ and the standard continuous-time surplus dynamics used in ruin analysis. The interarrival times $T_1, T_2, \dots$ are defined by $T_0 = 0$ and $T_k = t_k - t_{k-1}$ for $k > 0$, where t_k denotes the jump time of the k-th claim, and these interarrival variables are i.i.d. exponential random variables with rate λ. It follows that the n-th arrival jump time satisfies $t_n = \sum_{i=1}^{n} T_i$.

Cossette et al. [8] derived Lundberg's generalized equation of Cramér-Lundberg risk model for force of interest $\delta = 0$ is given by

$$\mathbb{E}\left[e^{r(cT - X)}\right] = 1, \tag{2}$$

the adjustment coefficient is denoted $-R$ with $R > 0$ and is defined as the non-zero root $r = R$ of the generalized Lundberg Eq. (2). Identifying the roots of this Eq. (2) is a critical prerequisite, as these roots characterize the singularities required to derive the explicit analytical expression for the ruin probability.

2.3 Farlie-Gumbel-Morgenstern Copula

We utilize the FGM copula specifically to introduce a dependence structure between claim sizes and inter-claim times while maintaining mathematical tractability, which is essential for obtaining closed-form solutions.

Definition 3 [14]. *A copula is a function $C: [0,1] \times [0,1] \to [0,1]$ that satisfies:*

a. $C(u, 0) = C(0, v) = 0$ *and* $C(u, 1) = u$; $C(1, v) = v$, *for all* $u, v \in [0,1]$.

b. $C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \ge 0$, *for any* $0 \le u_1 \le u_2 \le 1$ *and* $0 \le v_1 \le v_2 \le 1$.

The following presents Sklar's theorem, which lies at the core of copula theory and underpins many applications in statistical theory.

Theorem 1 [14]. *If H is a joint distribution with marginals F and G, then there exists a copula C such that $H(x,y) = C\big(F(x), G(y)\big)$ for all $x, y \in [-\infty, \infty]$. When F and G are continuous, the copula C is unique. Conversly, any copula C combined with marginals F and G via $H(x,y) = C\big(F(x), G(y)\big)$ yields a valid joint distribution with those marginals.*

For continuous marginals, the theorem extends to densities by

$$h(x,y) = c\big(F(x), G(y)\big) f(x) g(y), \tag{3}$$

where the copula density is $c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$, thus cleanly separating marginal behavior from dependence through C. As canonical example, the Farlie-Gumbel-Morgenstern (FGM) copula is given by

$$C(u,v) = uv\big(1 + \theta(1-u)(1-v)\big), \tag{4}$$

with $\theta \in [-1,1]$. FGM copula reduces to independence at $\theta = 0$, which matches the product copula for independent margins. Differentiating C yields the copula density

$$c(u,v) = 1 + \theta(1-2u)(1-2v). \tag{5}$$

The FGM copula is selected primarily for its mathematical tractability. Unlike Archimedean copulas (e.g., Clayton, Gumbel, or Frank) which model stronger dependence but complicate analytical derivations, the FGM copula's polynomial structure allows us to derive an explicit closed-form solution for the ruin probability using Laplace transforms. This analytical tractability is essential for the specific objectives of this study.

2.4 Modification of Rouche's Theorem

Theorem 2 [15]. *Let $f(z)$ and $g(z)$ be analytic function in the open disk $\{|z| < 1\}$ and continuous on the boundary $\{|z| < 1\}$, and assume that $|f(z)| > |g(z)|$ for all boundary points with $z \neq 1$, while $f(1) = -g(1) \neq 0$ holds at $z = 1$. If $f(z)$ and $g(z)$ differentiable at $z = 1$ and satisfy $\frac{f'(1)+g'(1)}{f(1)} > 0$, then number of zeros of $f(z) + g(z)$ in $|z| < 1$ equals to number of zeros of $f(z)$ in $|z| < 1$ minus one, i.e. $Z_{f+g} = Z_f - 1$.*

This theorem is a modified version of Rouche's theorem that applies when the usual sufficient condition $|f(z)| > |g(z)|$ does not hold at $z = 1$. We apply Theorem 2 to rigorously verify the existence and number of roots in the right half-plane, ensuring the validity of the partial fraction decomposition used in the final derivation.

2.5 Laplace Transform

This integral transform is the primary mathematical tool employed in this study to convert the complex integro-differential equation of ruin probability into a solvable algebraic equation.

Definition 4 [16]. *Assume there exists $c_0 \in \mathbb{R}$ such that $\int_0^\infty e^{-c_0 t}|f(t)|dt < \infty$. The Laplace transform of $f(t)$, denoted $\mathcal{L}\{f(t)\}(s) = f^\star(s)$, is defined by $f^\star(s) = \int_0^\infty e^{-st} f(t) dt$ for $s \in \mathbb{C}$ with $Re(s) \geq c_0$.*

Some key properties of the Laplace transform include linearity, transforms of derivatives and the convolution theorem:

a. Linearity: for constants a, b and transform $\mathcal{L}\{f(t)\}(s) = f^\star(s)$ and $\mathcal{L}\{g(t)\}(s) = g^\star(s)$, satisfies $\mathcal{L}\{af(t) + bg(t)\}(s) = af^\star(s) + bg^\star(s)$, which follows directly from linearity of the integral.

b. Derivatives: $\mathcal{L}\{f'(t)\}(s) = sf^{\star}(s) - f(0)$ and, more generally, $\mathcal{L}\{f^{(n)}(t)\}(s) = s^n f^{\star}(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$.
c. Convolution: if $(f * g)(t) = \int_0^t f(v)g(t-v)dv$, then $\mathcal{L}\{(f * g)(t)\}(s) = f^{\star}(s)g^{\star}(s)$.

2.6 Lagrange Interpolation

Lagrange interpolation is a polynomial interpolation method that constructs the unique polynomial of lowest degree that passes exactly through a given set of $n+1$ data points (x_i, y_i) for $i = 0, 1, \dots, n$.

Definition 5 [17]. *Given distinct nodes $x_0, \dots, x_n$ with values $y_0, \dots, y_n$, the Lagrange interpolation is $f(x) = \sum_{k=0}^{n} y_k L_k(x)$, where the Lagrange basis functions are $L_k(x) = \prod_{j \neq k} \frac{x - x_j}{x_k - x_j}$ for $k = 0, 1, \dots, n$.*

This polynomial $f(x)$ uniquely interpolates the given data and provides the desired estimate of the dependent variable at the target abscissa x. In our derivation, this interpolation technique is specifically applied to reconstruct the numerator polynomial of the ruin probability's Laplace transform based on the roots identified from Lundberg's equation.

3. RESULT AND DISCUSSION

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that accommodates all events and stochastics processes employed in this paper.

3.1 Definition of the Risk Model and the Dependence Structure based on FGM Copula

3.1.1 The Two Types of Claims Risk Model

We consider a classical risk model extended to accommodate two distinct categories of claims, denoted as type-I and type-II. These types are differentiated by their frequency and severity characteristics:

a. Type-I claims: represented by the claim counting process $\{N_1(t), t \geq 0\}$ with intensity $\lambda_1 > 0$ and claim sizes $\{X_i\}_{i \geq 1}$ having a cumulative distribution function (c.d.f.) F_X. This type typically characterizes high-frequency, low-severity risks.
b. Type-II claims: represented by the claim counting process $\{N_2(t), t \geq 0\}$ with intensity $\lambda_2 > 0$ and claim sizes $\{Y_j\}_{j \geq 1}$ having a c.d.f. F_Y. This type typically characterizes low-frequency, high-severity risks.

Let $S_1(t) = \sum_{i=1}^{N_1(t)} X_i$ and $S_2(t) = \sum_{j=1}^{N_2(t)} Y_j$ denote the aggregate claim amounts for type-I and type-II, respectively. The insurer's surplus process at time $t \geq 0$ denote by $U(t)$, is defined as

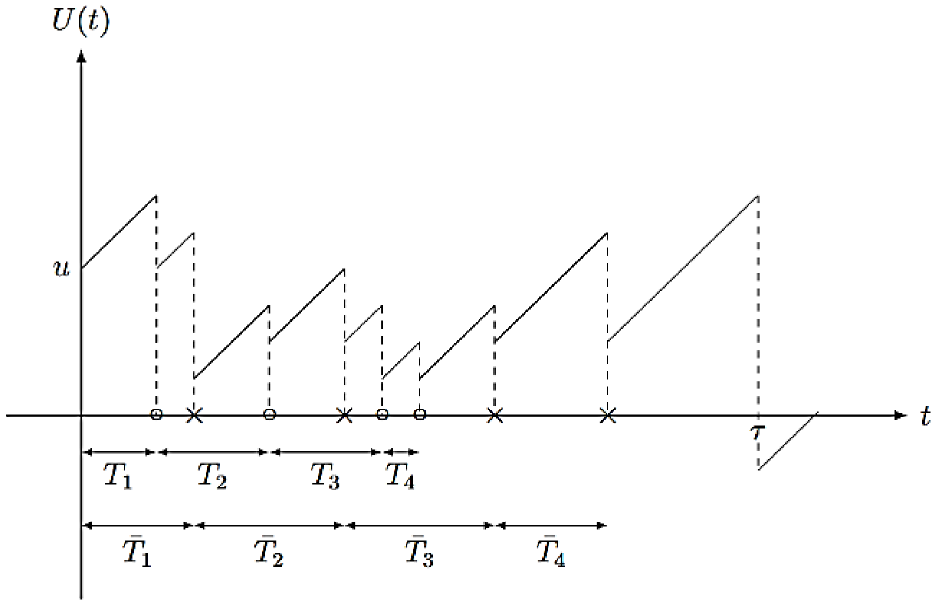
$$U(t) = u + ct - S_1(t) - S_2(t) = u + ct - \sum_{i=1}^{N_1(t)} X_i - \sum_{j=1}^{N_2(t)} Y_j, \tag{6}$$

where $u = U(0) \geq 0$ is the initial surplus, and premiums are collected continuously at rate $c > 0$. Eq. (6) illustrates the dynamic behavior of the surplus. The capital increases continuously due to premium income ct. However, it decreases due to two simultaneous claim processes with different behaviors, type-I claims cause frequent, small drops in the surplus, while type-II claims cause rare but large drops. The inter-arrival times corresponding to the claim counting processes are defined as follows:

a. Type-I inter-arrival times: let T_k denote the time elapsed between the $(k-1)$-th and the k-th claim of type-I. The sequence $\{T_k\}_{k \geq 1}$ consists of independent and identically distributed (i.i.d.) exponential random variables with rate λ_1.

b. Type-II inter-arrival times: let $\bar{T}_k$ denote the time elapsed between the $(k-1)$-th and the k-th claim of type-II. The sequence $\{\bar{T}_k\}_{k\geq 1}$ consists of i.i.d. exponential random variables with rate λ_2.

Regarding the independence assumptions, the processes governing type-I claims $(\{N_1(t)\}, \{X_i\})$ are assumed to be mutually independent of those governing type-II claims $\left(\{N_2(t)\}, \{Y_j\}\right)$. However, consistent with the objective of this study, we allow for a dependence structure within each claim type, specifically



between the claim size and its inter-arrival time which is modeled in the following subsection. An illustrative sample path of the surplus $U(t)$ is shown in the Figure 1.

Figure 1. Illustrative sample path of the surplus $U(t)$

We denote the time of ruin by τ. This variable represents the first time the insurer's surplus drops below zero. Its mathematical form is given by

$$\tau = \min\{t: t \geq 0, U(t) < 0\}. \tag{7}$$

If the surplus remains non-negative for all $t \geq 0$, then $\tau = \infty$. The ruin probability given initial surplus u is denoted by $\psi(u)$ and defined as

$$\psi(u) = \mathbb{P}(\tau < \infty | U(0) = u). \tag{8}$$

Observe that $N_1(t)$ and $N_2(t)$ denote the numbers of type-I and type-II claims in the interval $(0, t]$, respectively. Let $N(t)$ be the total number of claims of either type, so that $N(t) = N_1(t) + N_2(t)$. $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$. Since $\{N(t), t \geq 0\}$ is a Poisson process, the probability of more than one claim occurring in a short interval of length $h > 0$ is $o(h)$. This mathematical property implies that type-I and type-II claims almost surely do not occur simultaneously. Consequently, the event of ruin is triggered by a single claim belonging uniquely to either type-I or type-II, making these two ruin events mutually exclusive. Because of this mutual exclusivity and not merely due to the independence of the arrival processes, the total ruin probability can be defined as the sum of the individual probabilities

$$\psi(u) = \psi_1(u) + \psi_2(u), \tag{9}$$

with $0 \leq \psi_k(u) \leq 1$, where $\psi_k(u)$ denotes the probability of ruin caused by a claim of type k, for $k \in \{1, 2\}$.

3.1.2 The Dependence Structure

In this paper, the type-I claim sizes $\{X_i\}$ and their interarrival times $\{T_i\}$ are not assumed independent, their dependence is modeled via a copula. Similarly, the type-II claim sizes $\{Y_j\}$ are dependent on their interarrival times $\{\bar{T}_j\}$. Consider the i.i.d. sequence of random vectors $\{(X_i, T_i)\}_{i\geq 1}$ for each $i \geq 1$, the dependence between X_i and T_i is modeled using the FGM copula with parameter $\theta_1 \in [-1,1]$. Likewise, the i.i.d. sequence $\{(Y_j, \bar{T}_j)\}_{j\geq 1}$ is such that, for each $j \geq 1$. The dependence between Y_j and $\bar{T}_j$ is modeled using the FGM copula with parameter $\theta_2 \in [-1,1]$. This assumption implies that the size of the i-th claim depends only on the interarrival time between the $(i-1)$-th and the i-th claims.

By Sklar's theorem (see Theorem 1), the joint c.d.f. of (X_i, T_i) is

$$\begin{aligned} F_{X,T}(x,t) &= C\big(F_X(x), F_T(t)\big) \\ &= F_X(x)F_T(t) + \theta_1 F_X(x)F_T(t)\big(1-F_X(x)\big)\big(1-F_T(t)\big). \end{aligned} \tag{10}$$

Using Eq. (3) and Eq. (5), the joint p.d.f of (X_i, T_i) is given by

$$\begin{aligned} f_{X,T}(x,t) &= c\big(F_X(x), F_T(t)\big)f_X(x)f_T(t) \\ &= \big[1+\theta_1\big(1-2F_X(x)\big)\big(1-2F_T(t)\big)\big]\, f_X(x)f_T(t) \\ &= f_X(x)f_T(t) + \theta_1 f_X(x)f_T(t)\big(1-2F_X(x)\big)\big(1-2F_T(t)\big). \end{aligned} \tag{11}$$

Let $h_X(x) := f_X(x)\big(1-2F_X(x)\big)$. Since T_i are i.i.d. exponential with rate λ_1, we have $F_T(t) = 1 - e^{-\lambda_1 t}$, and $f_T(t) = \lambda_1 e^{-\lambda_1 t}$. From Eq. (11), we have

$$f_{X,T}(x,t) = \lambda_1 e^{-\lambda_1 t} f_X(x) + \theta_1 h_X(x)\big(2\lambda_1 e^{-2\lambda_1 t} - \lambda_1 e^{-\lambda_1 t}\big), \tag{12}$$

for $x \geq 0, t \geq 0$. From Eq. (12), the conditional joint p.d.f. of the bivariate (X_i, T_i) is given by

$$\begin{aligned} f_{X,T|T<\bar{T}}(x,t) &= \frac{f_{X,T}(x,t)\mathbb{P}(\bar{T} > t)}{\mathbb{P}(T < \bar{T})} \\ &= \frac{(\lambda_1+\lambda_2)e^{-\lambda_2 t} f_{X,T}(x,t)}{\lambda_1}. \end{aligned} \tag{13}$$

By a similar argument as above, the joint p.d.f. of $(Y_j, \bar{T}_j)$ and the conditional joint p.d.f. of the bivariate $(Y_j, \bar{T}_j)$ is given by

$$f_{Y,\bar{T}}(y,t) = \lambda_2 e^{-\lambda_2 t} f_Y(y) + \theta_2 h_Y(x)\big(2\lambda_2 e^{-2\lambda_2 t} - \lambda_2 e^{-\lambda_2 t}\big), \tag{14}$$

$$f_{X,T|T<\bar{T}}(x,t) = \frac{(\lambda_1+\lambda_2)e^{-\lambda_1 t} f_{Y,\bar{T}}(y,t)}{\lambda_2}, \tag{15}$$

for $y \geq 0, t \geq 0$, where $h_Y(y) := f_Y(y)\big(1-2F_Y(y)\big)$.

To ensure the insurer is almost surely solvent, the solvability condition must hold for the entire portfolio. Using the law of total expectation, the premium rate c is determined such that the expected premium income exceeds the expected aggregate claim amounts from both type-I and type-II. This condition is expressed as

$$\mathbb{P}(T < \bar{T})\mathbb{E}[cT - X | T < \bar{T}] + \mathbb{P}(\bar{T} < T)\mathbb{E}[c\bar{T} - Y | \bar{T} < T] > 0. \tag{16}$$

3.2 Lundberg's Equation

This section derives the Lundberg's equation for the two types of claim risk process. This single equation characterizes the aggregate risk by unifying the contributions of both claim types. Using the law of total expectation, we condition on the first claim occurrence to distinguish between the two types. Consequently, the equation is constructed as the sum of two components, the first term corresponds to the scenario where a type-I claim occurs first, and the second term corresponds to a type-II claim occurring first. This relationship is expressed as

$$\mathbb{P}(T<\bar{T})\mathbb{E}\left[e^{s(cT-X)}\middle|T<\bar{T}\right]+\mathbb{P}(\bar{T}<T)\mathbb{E}\left[e^{s(c\bar{T}-Y)}\middle|\bar{T}<T\right]=1. \tag{17}$$

Since $T\sim Exp(\lambda_1)$ and $\bar{T}\sim Exp(\lambda_2)$ are independent, it follows that $\mathbb{P}(T<\bar{T})=\frac{\lambda_1}{\lambda_1+\lambda_2}$ and $\mathbb{P}(\bar{T}<T)=\frac{\lambda_2}{\lambda_1+\lambda_2}$.

Given in Eqs. (12)-(15), the left-hand side of Eq. (17) can be written as

$$\begin{aligned}&\mathbb{P}(T<\bar{T})\mathbb{E}\left[e^{s(cT-X)}\middle|T<\bar{T}\right]+\mathbb{P}(\bar{T}<T)\mathbb{E}\left[e^{s(c\bar{T}-Y)}\middle|\bar{T}<T\right]\\&=\frac{\lambda_1}{\lambda_1+\lambda_2}\left[\int_0^\infty\int_0^\infty e^{sct}e^{-sx}\left[\lambda_1e^{-\lambda_1t}f_X(x)+\theta_1h_X(x)\left(2\lambda_1e^{2\lambda_1t}-\lambda_1e^{-\lambda_1t}\right)\right]e^{-\lambda_2t}dxdt\right]\\&+\frac{\lambda_2}{\lambda_1+\lambda_2}\left[\int_0^\infty\int_0^\infty e^{sct}e^{-sy}\left[\lambda_2e^{-\lambda_2t}f_Y(y)+\theta_2h_Y(y)\left(2\lambda_2e^{2\lambda_2t}-\lambda_2e^{-\lambda_2t}\right)\right]e^{-\lambda_1t}dydt\right].\end{aligned} \tag{18}$$

Combining Eqs. (17) and (18), we obtain

$$\begin{aligned}&\frac{\lambda_1}{\lambda_1+\lambda_2}\left\{\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2-sc}f_X^\star(s)+\theta_1h_X^\star(s)\left[\frac{2(\lambda_1+\lambda_2)}{2\lambda_1+\lambda_2-sc}-\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2-sc}\right]\right\}\\&+\frac{\lambda_2}{\lambda_1+\lambda_2}\left\{\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2-sc}f_Y^\star(s)+\theta_2h_Y^\star(s)\left[\frac{2(\lambda_1+\lambda_2)}{\lambda_1+2\lambda_2-sc}-\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2-sc}\right]\right\}\\&=1,\end{aligned} \tag{19}$$

with $f_X^\star(s)$, $f_Y^\star(s)$, $h_X^\star(s)$, $h_Y^\star(s)$ denoting the Laplace transforms of $f_X(x)$, $f_Y(y)$, $h_X(x)$, $h_Y(y)$, respectively.

Proposition 1 *Consider the Lundberg's equation given in Eq. (19). If* $\theta_1\neq0$ *and* $\theta_2\neq0$*, then Eq. (19) has three roots* $\varrho_1,\varrho_2,\varrho_3$ *with* $Re(\varrho_i)>0$ *for* $i=1,2$*, and one root equal to 0, namely* $\varrho_3=0$.

Proof. The proof begins by rewriting the Lundberg's equation in Eq. (19) into the form

$$\begin{aligned}&\frac{\lambda_1}{\lambda_1+\lambda_2-sc}\left(f_X^\star(s)+\theta_1h_X^\star(s)\right)\left[\frac{2\lambda_1}{2\lambda_1+\lambda_2-sc}-\frac{\lambda_1}{\lambda_1+\lambda_2-sc}\right]\\&+\frac{\lambda_2}{\lambda_1+\lambda_2-sc}\left(f_Y^\star(s)+\theta_2h_Y^\star(s)\right)\left[\frac{2\lambda_2}{\lambda_1+2\lambda_2-sc}-\frac{\lambda_2}{\lambda_1+\lambda_2-sc}\right]=1.\end{aligned} \tag{20}$$

Multiplying both sides by the polynomial $(\lambda_1+\lambda_2-sc)(2\lambda_1+\lambda_2-sc)(\lambda_1+2\lambda_2-sc)$, we obtain

$$\vartheta_1(s)+\vartheta_2(s)=0, \tag{21}$$

where $\vartheta_1(s)=\lambda_1(2\lambda_1+\lambda_2-sc)(\lambda_1+2\lambda_2-sc)f_X^\star(s)+\theta_1h_X^\star(s)[2\lambda_1(\lambda_1+\lambda_2-sc)(\lambda_1+2\lambda_2-sc)-\lambda_1(2\lambda_1+\lambda_2-sc)(\lambda_1+2\lambda_2-sc)]+\lambda_2(2\lambda_1+\lambda_2-sc)(\lambda_1+2\lambda_2-sc)f_Y^\star(s)+\theta_2h_Y^\star(s)[2\lambda_2(\lambda_1+\lambda_2-sc)(2\lambda_1+\lambda_2-sc)-\lambda_2(2\lambda_1+\lambda_2-sc)(\lambda_1+2\lambda_2-sc)]$, and $\vartheta_2(s)=-(\lambda_1+\lambda_2-sc)(2\lambda_1+\lambda_2-sc)(\lambda_1+2\lambda_2-sc)$. Both $\vartheta_1(s)$ and $\vartheta_2(s)$ are analytic in the right half-plane (except possibly at $s=0$) and continuous on its boundary, since Laplace transforms are analytic for $Re(s)>0$ and polynomials are entire. By considering limiting domain $D=\lim_{k\to\infty}\left\{s:\left|1-\frac{s}{k}\right|=1\right\}$, applying Theorem 2 and solvability condition in (16), it follows that the equation $\vartheta_1(s)+\vartheta_2(s)=0$ has the same number of roots inside D as $\vartheta_2(s)=0$ minus one. As $\vartheta_2(s)$ has three roots, Eq. (21) has exactly two roots with $Re(\varrho_i)>0$, for $i=1,2$ and one trivial root is zero. This completes the proof. ■

3.3 Integro-differential Equation

By Eq. (9) the ruin probability can be written as the sum of the probabilities of ruin caused type-I and type-II claims, $\psi(u)=\psi_1(u)+\psi_2(u)$. This section's goal is to formulate an integro-differential equation for the ruin probability caused type-I claims, $\psi_1(u)$ and type-II claims, $\psi_2(u)$. To calculate $\psi_1(u)$, by conditioning on the time and the amount of the first claim, there are four different possible scenarios:

a. a type-I claim of size x with $x \leq u + ct$,
b. a type-I claim of size x with $x > u + ct$, in which case $\psi_1(u) = 1$,
c. a type-II claim of size y with $y \leq u + ct$,
d. a type-II claim of size y with $y > u + ct$, in which case $\psi_2(u) = 0$.

Considering these four scenarios, we have

$$\begin{aligned}\psi_1(u) = \mathbb{P}(T < \bar{T})\bigg[&\int_0^\infty \int_0^{u+ct} \psi_1(u + ct - x) f_{X,T|T<\bar{T}}(x,t)dxdt \\ &+ \int_0^\infty \int_{u+ct}^\infty f_{X,T|T<\bar{T}}(x,t)dxdt\bigg] \\ &+ \mathbb{P}(\bar{T} < T)\int_0^\infty \int_0^{u+ct} \psi_1(u + ct - y) f_{Y,\bar{T}|\bar{T}<T}(y,t)dydt.\end{aligned} \tag{22}$$

Eq. (22) comprises three integral terms corresponding to the non-zero scenarios. The first term represents scenario (a), where a type-I claim occurs ($T < \bar{T}$) but does not cause ruin. The second term represents scenario (b), where a type-I claim causes immediate ruin, the probability is 1, leaving only the density function. The third term represents scenario (c), where a type-II claim occurs ($\bar{T} < T$) but does not cause ruin. Note that scenario (d) does not appear in the equation because if ruin is caused by a type-II claim, the probability of ruin caused by type-I is zero ($\psi_1(u) = 0$), causing the term to vanish.

Let

$$\begin{aligned} &v = u + ct, \\ &\sigma_1(v) = \int_0^v \psi_1(v - x) f_X(x)dx + m_1(v);\ m_1(v) = \int_v^\infty f_X(x)dx, \\ &\sigma_2(v) = \int_0^v \psi_1(v - x) h_X(x)dx + m_2(v);\ m_2(v) = \int_v^\infty h_X(x)dx, \\ &\sigma_3(v) = \int_0^v \psi_1(v - y) f_Y(y)dy,\ \sigma_4(v) = \int_0^v \psi_1(v - y) h_Y(y)dy. \end{aligned} \tag{23}$$

Given from Eqs. (13), (12), and (23), Eq. (22) becomes

$$\begin{aligned}\psi_1(u) = &\frac{\lambda_1}{c}\int_u^\infty e^{-(\lambda_1+\lambda_2)\left(\frac{v-u}{c}\right)}\sigma_1(v)dv + \frac{2\lambda_1\theta_1}{c}\int_u^\infty e^{-(2\lambda_1+\lambda_2)\left(\frac{v-u}{c}\right)}\sigma_2(v)dv \\ &-\frac{\lambda_1\theta_1}{c}\int_u^\infty e^{-(\lambda_1+\lambda_2)\left(\frac{v-u}{c}\right)}\sigma_2(v)dv + \frac{\lambda_2}{c}\int_u^\infty e^{-(\lambda_1+\lambda_2)\left(\frac{v-u}{c}\right)}\sigma_3(v)dv \\ &+\frac{2\lambda_2\theta_2}{c}\int_u^\infty e^{-(\lambda_1+2\lambda_2)\left(\frac{v-u}{c}\right)}\sigma_4(v)dv - \frac{\lambda_2\theta_2}{c}\int_u^\infty e^{-(\lambda_1+\lambda_2)\left(\frac{v-u}{c}\right)}\sigma_4(v)dv.\end{aligned} \tag{24}$$

Let $G_1(u) = \left(\frac{\lambda_1+\lambda_2}{c}\right)\psi_1(u) - \psi_1'(u)$, where $\psi_1'(u)$ denotes the derivative of $\psi_1(u)$ with respect to u, then it follows that we obtain

$$\begin{aligned}G_1(u) = &\frac{2\lambda_1\theta_1}{c}\left(\frac{\lambda_1+\lambda_2}{c} - \frac{2\lambda_1+\lambda_2}{c}\right)\int_u^\infty e^{-(2\lambda_1+\lambda_2)\left(\frac{v-u}{c}\right)}\sigma_2(v)dv \\ &+\frac{2\lambda_2\theta_2}{c}\left(\frac{\lambda_1+\lambda_2}{c} - \frac{\lambda_1+2\lambda_2}{c}\right)\int_u^\infty e^{-(\lambda_1+2\lambda_2)\left(\frac{v-u}{c}\right)}\sigma_4(v)dv \\ &+\frac{\lambda_1}{c}\sigma_1(u) + \frac{\lambda_1\theta_1}{c}\sigma_2(u) + \frac{\lambda_2}{c}\sigma_3(u) + \frac{\lambda_2\theta_2}{c}\sigma_4(u).\end{aligned} \tag{25}$$

Next, suppose $G_2(u) = \left(\frac{2\lambda_1+\lambda_2}{c}\right) G_1(u) - G_1'(u)$, where $G_1'(u)$ denotes the derivative of $G_1(u)$ with respect to u, it follows that we obtain

$$\begin{aligned} G_2(u) = &-\frac{2\lambda_2^2\theta_2}{c^2}\left(\frac{\lambda_1-\lambda_2}{c}\right)\int_u^\infty e^{-(\lambda_1+2\lambda_2)\left(\frac{v-u}{c}\right)}\sigma_4(v)dv + \frac{\lambda_1}{c}\left(\frac{2\lambda_1+\lambda_2}{c}\right)\sigma_1(u) \\ &+\frac{\lambda_1\lambda_2\theta_1}{c}\sigma_2(u) + \frac{\lambda_2}{c}\left(\frac{2\lambda_1+\lambda_2}{c}\right)\sigma_3(u) + \frac{\lambda_2\theta_2}{c}\left(\frac{2\lambda_1-\lambda_2}{c}\right)\sigma_4(u) - \frac{\lambda_1}{c}\sigma_1'(u) \\ &-\frac{\lambda_1\theta_1}{c}\sigma_2'(u) - \frac{\lambda_2}{c}\sigma_3'(u) - \frac{\lambda_2\theta_2}{c}\sigma_4'(u). \end{aligned} \tag{26}$$

From Eq. (26), differentiate $G_2(u)$ with respect to u, and thus we obtain

$$\begin{aligned} \left(\frac{\lambda_1+2\lambda_2}{c}\right)&G_2(u) - G_2'(u) \\ &= C_{\sigma_1}\sigma_1(u) - C_{\sigma_1'}\sigma_1'(u) + C_{\sigma_1''}\sigma_1''(u) + C_{\sigma_2}\sigma_2(u) - C_{\sigma_2'}\sigma_2'(u) \\ &+ C_{\sigma_2''}\sigma_2''(u) + C_{\sigma_3}\sigma_3(u) - C_{\sigma_3'}\sigma_3'(u) + C_{\sigma_3''}\sigma_3''(u) + C_{\sigma_4}\sigma_4(u) \\ &- C_{\sigma_4'}\sigma_4'(u) + C_{\sigma_4''}\sigma_4''(u), \end{aligned} \tag{27}$$

where

$$\begin{aligned} C_{\sigma_1} &= \frac{\lambda_1(2\lambda_1+\lambda_2)(\lambda_1+2\lambda_2)}{c^3}, C_{\sigma_1'} = \frac{3\lambda_1(\lambda_1+\lambda_2)}{c^2}, C_{\sigma_1''} = \frac{\lambda_1}{c}, \\ C_{\sigma_2} &= \frac{\lambda_1\lambda_2\theta_1(\lambda_1+2\lambda_2)}{c^3}, C_{\sigma_2'} = \frac{\lambda_1\theta_1(\lambda_1+3\lambda_2)}{c^2}, C_{\sigma_2''} = \frac{\lambda_1\theta_1}{c}, \\ C_{\sigma_3} &= \frac{\lambda_2(2\lambda_1+\lambda_2)(\lambda_1+2\lambda_2)}{c^3}, C_{\sigma_3'} = \frac{3\lambda_2(\lambda_1+\lambda_2)}{c^2}, C_{\sigma_3''} = \frac{\lambda_2}{c}, \\ C_{\sigma_4} &= \frac{\lambda_1\lambda_2\theta_2(2\lambda_1+\lambda_2)}{c^3}, C_{\sigma_4'} = \frac{\lambda_2\theta_2(3\lambda_1+\lambda_2)}{c^2}, C_{\sigma_4''} = \frac{\lambda_2\theta_2}{c}. \end{aligned} \tag{28}$$

The left-hand side of Eq. (27) can be expressed into $\psi(u)$ terms as follows

$$\left(\frac{\lambda_1+2\lambda_2}{c}\right)G_2(u) - G_2'(u) = C_{\psi_1}\psi_1(u) - C_{\psi_1'}\psi_1'(u) + C_{\psi_1''}\psi_1''(u) - \psi_1'''(u), \tag{29}$$

with

$$\begin{aligned} C_{\psi_1} &= \frac{(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2)(\lambda_1+2\lambda_2)}{c^3}, C_{\psi_1'} = \frac{(2\lambda_1+\lambda_2)(\lambda_1+2\lambda_2)+3(\lambda_1+\lambda_2)^2}{c^2}, \\ C_{\psi_1''} &= \frac{4(\lambda_1+\lambda_2)}{c}. \end{aligned} \tag{30}$$

Combining Eqs. (27) and (29), we obtain

$$\begin{aligned} C_{\psi_1}&\psi_1(u) - C_{\psi_1'}\psi_1'(u) + C_{\psi_1''}\psi_1''(u) - \psi_1'''(u) \\ &= C_{\sigma_1}\sigma_1(u) - C_{\sigma_1'}\sigma_1'(u) + C_{\sigma_1''}\sigma_1''(u) + C_{\sigma_2}\sigma_2(u) - C_{\sigma_2'}\sigma_2'(u) \\ &+ C_{\sigma_2''}\sigma_2''(u) + C_{\sigma_3}\sigma_3(u) - C_{\sigma_3'}\sigma_3'(u) + C_{\sigma_3''}\sigma_3''(u) + C_{\sigma_4}\sigma_4(u) \\ &- C_{\sigma_4'}\sigma_4'(u) + C_{\sigma_4''}\sigma_4''(u). \end{aligned} \tag{31}$$

3.4 Laplace Transform of Ruin Probability

Solving the integro-differential equation in Eq. (31) directly is analytically challenging due to the presence of convolution terms and high-order derivatives. To overcome this complexity, we apply the Laplace transform with respect to the initial surplus u. The application of this transform is mathematically valid because the ruin probability $\psi_k(u)$ is a bounded function defined on $[0,\infty)$, which ensures the convergence of the integral. The primary aim of applying the Laplace transform is to convert the complex integro-differential equation into a simpler algebraic equation in the s-domain. This allows us to solve for $\psi_k^\star(s)$ explicitly before inverting it back to finding the solution $\psi_k(u)$. Based on this approach, the result is stated in the following proposition.

Proposition 2 *In the two types of claims risk model with dependence structure by FGM copula, the Laplace transform of the ruin probability caused type-k claims $\psi_k(u)$ is given by*

$$\psi_k^\star = \frac{B_1^{(k)}(s) + B_2^{(k)}(s)}{c^{-3}\big(-\vartheta_1(s) - \vartheta_2(s)\big)}, \tag{32}$$

where $\vartheta_1(s)$ and $\vartheta_2(s)$ are those defined in Eq. (21). The function $B_1^{(k)}(s)$ is

$$B_1^{(k)}(s) = \left(C_{\sigma_{2k-1}} - C_{\sigma'_{2k-1}} s + C_{\sigma''_{2k-1}} s^2\right) m_{2k-1}^\star(s) + \left(C_{\sigma_{2k}} - C_{\sigma'_{2k}} s + C_{\sigma''_{2k}} s^2\right) m_{2k}^\star(s), \tag{33}$$

and $B_2^{(k)}(s)$ is the polynomial in s is given by

$$B_2^{(k)}(s) = -\sum_{j=1}^{3} B_1^{(k)}\left(\varrho_j\right)\left(\prod_{k=1,k\neq j}^{3} \frac{s - \varrho_k}{\varrho_j - \varrho_k}\right), \tag{34}$$

with $\varrho_1, \varrho_2, \varrho_3$ denoting the three roots of the Lundberg's equation.

Proof. The proof of the proposition is provided only for the ruin probability caused type-I claims. Using properties of the Laplace transform, taking the Laplace transform on both side in Eq. (31), and isolate $\psi_1^\star(s)$, we obtain

$$\psi_1^\star(s) = \frac{\mathcal{N}_1(s)}{\mathcal{D}_1(s)} \tag{35}$$

where

$$\begin{aligned} \mathcal{N}_1(s) = {} & \left(C_{\sigma_1} - C_{\sigma'_1} s + C_{\sigma''_1} s^2\right) m_1^\star(s) + \left(C_{\sigma_2} - C_{\sigma'_2} s + C_{\sigma''_2} s^2\right) m_2^\star(s) \\ & + \left(C_{\sigma'_1} - C_{\sigma''_1} s\right) m_1(0) + \left(C_{\sigma'_2} - C_{\sigma''_2} s\right) m_2(0) - C_{\sigma''_1} m'_1(0) - C_{\sigma''_2} m'_2(0) \\ & - \Big[\Big(C_{\sigma''_1} f_X(0) + C_{\sigma''_2} h_X(0) + C_{\sigma''_3} f_Y(0) + C_{\sigma''_4} h_Y(0) + C_{\psi'_1} - C_{\psi''_1} s + s^2\Big)\psi_1(0) \\ & \qquad + \left(s - C_{\psi''_1}\right)\psi'_1(0) + \psi''_1(0)\Big], \end{aligned} \tag{36}$$

$$\begin{aligned} \mathcal{D}_1(s) = {} & C_{\psi_1} - C_{\sigma_1} f_X^\star(s) - C_{\sigma_2} h_X^\star(s) - C_{\sigma_3} f_Y^\star(s) - C_{\sigma_4} h_Y^\star(s) \\ & + \Big[C_{\sigma'_1} f_X^\star(s) + C_{\sigma'_2} h_X^\star(s) + C_{\sigma'_3} f_Y^\star(s) + C_{\sigma'_4} h_Y^\star(s) - C_{\psi'_1}\Big] s \\ & + \Big[C_{\psi''_1} - C_{\sigma''_1} f_X^\star(s) - C_{\sigma''_2} h_X^\star(s) - C_{\sigma''_3} f_Y^\star(s) - C_{\sigma''_4} h_Y^\star(s)\Big] s^2 - s^3. \end{aligned} \tag{37}$$

Note that the Lundberg's Eq. (21) can be written as $c^3\mathcal{D}_1(s) = 0$. By means of Proposition 1, the denominator of Eq. (35) has three roots $\varrho_1, \varrho_2, \varrho_3$. Due to the analyticity of the numerator of Eq. (35), it requires that these are also roots of the numerator. Futhermore, we define $\mathcal{N}_1(s) = B_1^{(1)}(s) + B_2^{(1)}(s)$, such that $B_1^{(1)}(s)$ is the sum of all terms that include $m_1^\star(s)$ and $m_2^\star(s)$ and $B_2^{(1)}(s)$ is the sum of the remaining terms. It is found that $B_2^{(1)}(s)$ is a polynomial of degree two. Since $\mathcal{N}_1(\varrho_i) = 0$, for $i = 1, 2, 3$, it can be written $B_2^{(1)}(\varrho_i) = -B_1^{(1)}(\varrho_i)$. By using the Lagrange interpolation formula (see Definition 5) at the three points $\varrho_1, \varrho_2, \varrho_3$ we obtain Eq. (34). For the Laplace transform of $\psi_2^\star(s)$, the derivation follows the same analogy for $\psi_1^\star(s)$. This completes the proof. ■

Thus, by using the linearity property of Laplace transform and Proposition 2, it follows that $\psi^\star(s) = \psi_1^\star(s) + \psi_2^\star(s)$. To revert the solution from the Laplace domain back to the original surplus domain, we observe that the resulting expression $\psi_k^\star(s)$ for is a rational function or a ratio of polynomials. Consequently, the inversion is performed by applying partial fraction decomposition to expand $\psi_k^\star(s)$ into a sum of elementary terms. The explicit solution $\psi_k(u)$ is then obtained by applying the inverse Laplace transform term by term. This inversion process is demonstrated explicitly for the case of exponentially distributed claim sizes in Section 3.6.

3.5 Analysis of Ruin Probability when $u = 0$

This section we analyze of ruin probability by considering the case of $u = 0$. The roots of the Lundberg's equation, as discussed in section 3.2, are fundamental to the subsequent analysis. We assume the roots $\varrho_1, \varrho_2, \varrho_3$ are all distinct. Let $\mathcal{K} = C_{\sigma_1''} f_X(0) + C_{\sigma_2''} h_X(0) + C_{\sigma_3''} f_Y(0) + C_{\sigma_4''} h_Y(0)$. Note that ϱ_1, ϱ_2, and ϱ_3 are roots of $\mathcal{N}_1(s) = B_1^{(1)}(s) + B_2^{(1)}(s)$. From Eq. (36), the following holds

$$\begin{aligned}\left(\mathcal{K} + C_{\psi_1'} - C_{\psi_1''}\varrho_i + \varrho_i^2\right)\psi_1(0) + \left(\varrho_i - C_{\psi_1''}\right)\psi_1'(0) + \psi_1''(0) \\ = \sum_{j=1}^{2}\left[\left(C_{\sigma_j'} - C_{\sigma_j''}\varrho_i\right)m_j(0) - C_{\sigma_j''}m_j'(0) + \left(C_{\sigma_j} - C_{\sigma_j'}\varrho_i + C_{\sigma_j''}\varrho_i^2\right)m_1^\star(\varrho_i)\right],\end{aligned} \tag{38}$$

for $i = 1, 2, 3$. Let $J(\varrho_i)$ be the right-hand side of Eq. (38). Since the roots ϱ_1, ϱ_2, and ϱ_3 are all distinct, we can form a system of equations which is expressed in the following matrix equation

$$\begin{bmatrix}\left(\mathcal{K} + C_{\psi_1'} - C_{\psi_1''}\varrho_1 + \varrho_1^2\right) & \left(\varrho_1 - C_{\psi_1''}\right) & 1 \\ \left(\mathcal{K} + C_{\psi_1'} - C_{\psi_1''}\varrho_2 + \varrho_2^2\right) & \left(\varrho_2 - C_{\psi_1''}\right) & 1 \\ \left(\mathcal{K} + C_{\psi_1'} - C_{\psi_1''}\varrho_3 + \varrho_3^2\right) & \left(\varrho_3 - C_{\psi_1''}\right) & 1\end{bmatrix}\begin{bmatrix}\psi_1(0) \\ \psi_1'(0) \\ \psi_1''(0)\end{bmatrix} = \begin{bmatrix}J(\varrho_1) \\ J(\varrho_2) \\ J(\varrho_3)\end{bmatrix}. \tag{39}$$

Based on the Eq. (39), our objective is to determine the value of $\psi_1(0)$. This can be accomplished using several standard methods from linear algebra. One approach is to solve the entire system by finding the inverse of the coefficient matrix. Alternatively, Cramér's rule can be applied to directly compute the value for $\psi_1(0)$. For the $\psi_2(0)$, the derivation follows the same analogy for $\psi_1(0)$. From Eq. (9), it follows that $\psi(0) = \psi_1(0) + \psi_2(0)$.

3.6 Exponentially Distributed Claims

This section presents an analytical formula for the probability ruin $\psi(u)$. It is further assumed that type-I and type-II claim sizes are exponentially distributed, characterized by the c.d.f $F_X(x) = 1 - e^{-\alpha_1 x}$, $f_X(x) = \alpha_1 e^{-\alpha_1 x}$, the Laplace transform $f_X^\star(s) = \alpha_1(\alpha_1 + s)^{-1}$ and the c.d.f $F_Y(y) = 1 - e^{-\alpha_2 y}$, $f_Y(y) = \alpha_2 e^{-\alpha_2 y}$, the Laplace transform $f_Y^\star(s) = \alpha_2(\alpha_2 + s)^{-1}$. It follows that $h_X^\star(s) = \alpha_1 s\left((2\alpha_1 + s)(\alpha_1 + s)\right)^{-1}$ and $h_Y^\star(s) = \alpha_2 s\left((2\alpha_2 + s)(\alpha_2 + s)\right)^{-1}$. From Eq. (23), we obtain $m_1'(u) = -f_X(u)$, $m_1^\star(s) = (\alpha_1 + s)^{-1}$, $m_2'(u) = -h_X(u)$ and $m_2^\star(s) = -\alpha_1\left((2\alpha_1 + s)(\alpha_1 + s)\right)^{-1}$.

Proposition 3 *Let* $-R_j$, *for* $j = 1, \dots, 4$, *be the distinct roots with* $Re(R_j) > 0$. *Then the explicit expression for the ruin probability caused type-k claims* $\psi_k(u)$ *for* $u \geq 0$ *is given by*

$$\psi_k(u) = \sum_{j=1}^{4} \varpi_{k,j} e^{-Ru}, \tag{40}$$

where

$$\varpi_{k,j} = -\left[\mathcal{B}_1^{(k)}(-R_j) + \mathcal{B}_2^{(k)}(-R_j)\right]\prod_{k=1}^{3}\left(\frac{1}{-R_j - \varrho_k}\right)\prod_{k=1,k\neq j}^{4}\left(\frac{1}{-R_j + R_k}\right), \tag{41}$$

and

$$\mathcal{B}_1^{(k)}(s) + \mathcal{B}_2^{(k)}(s) = \left[B_1^{(k)}(s) + B_2^{(k)}(s)\right](2\alpha_1 + s)(\alpha_1 + s)(2\alpha_2 + s)(\alpha_2 + s). \tag{42}$$

Proof. First we substitute $f_X^\star(s), h_X^\star(s), f_Y^\star(s)$ and $h_Y^\star(s)$ into Lundberg's Eq. (21) to obtain an equation that is a seventh-degree polynomial. This Lundberg's equation has three roots, ϱ_i with $Re(\varrho_i) > 0$ for $i = 1, 2, 3$, and four additional roots, $-R_j$ with $Re(R_j) > 0$ for $j = 1, \dots, 4$. Consequently, the Lundberg's equation can be factored as

$$-c^3(s-\varrho_1)(s-\varrho_2)(s-\varrho_3)(s+R_1)(s+R_2)(s+R_3)(s+R_4)=0. \tag{43}$$

We then define the functions $\ell(s)$ as follows

$$\ell(s)=\left[c^{-3}\big(-\vartheta_1(s)-\vartheta_2(s)\big)\right](2\alpha_1+s)(\alpha_1+s)(2\alpha_2+s)(\alpha_2+s), \tag{44}$$

and the functions $\mathcal{B}_1^{(k)}(s)+\mathcal{B}_2^{(k)}(s)$ in Eq. (42). This gives the Laplace transform of the ruin probability caused type-I claims

$$\psi_1^{\star}(s)=\frac{\mathcal{B}_1^{(1)}(s)+\mathcal{B}_2^{(1)}(s)}{\ell(s)}, \tag{45}$$

The denominator of Eq. (45) can be expressed in terms of the roots of the Lundberg equation in Eq. (43). Using the Lagrange interpolation formula, the numerator of Eq. (45) can be written as

$$\mathcal{B}_1^{(1)}(s)+\mathcal{B}_2^{(1)}(s)=\sum_{i=1}^{4}\left\{\left[\mathcal{B}_1^{(1)}(-R_i)+\mathcal{B}_2^{(1)}(-R_i)\right]\prod_{k=1}^{3}\left(\frac{s-\varrho_k}{-R_j-\varrho_k}\right)\prod_{k=1,k\neq j}^{4}\left(\frac{s+R_k}{-R_j+R_k}\right)\right\}, \tag{46}$$

By substituting $\mathcal{B}_1^{(1)}(s)+\mathcal{B}_2^{(1)}(s)$ in Eq. (46) back into the Eq. (45), several terms cancel out, simplifying the expression to

$$\psi_1^{\star}(s)=\sum_{j=1}^{4}\frac{\varpi_{1,j}}{s+R_j}, \tag{47}$$

where $\varpi_{k,j}$ is defined in Eq. (41). Finally, applying the inverse Laplace transform term by term and utilizing the linearity property, we obtain the explicit expression for the ruin probability caused by type-I claims in the surplus domain $\psi_k(u)$. The probability ruin caused type-II claims is derived analogously to that of type-I claims. Hence, we can obtain the total ruin probability. ■

To illustrate, a numerical example is provided as follows.

Example 1. (Case of type-I claims occurring frequently but with a low average claim sizes, and type-II claims occurring rarely but with a high average claim sizes) Consider a risk model with two types of claims. The claim arrival processes for type-I, $N_1(t)$, and type-II, $N_2(t)$, follow Poisson processes with intensities $\lambda_1 = 1.0$ and $\lambda_2 = 0.2$, respectively. The claim sizes for each type are assumed to follow an exponential distribution. The type-I claim size, X, is exponentially distributed with parameter $\alpha_1 = 1.0$, and the type-II claim size, Y, is exponentially distributed with parameter $\alpha_2 = 0.2$. The dependence structure between the claim size and the inter-arrival time for each respective type is modeled by the FGM copula, with dependence parameters θ_1 and θ_2. These parameters are specifically chosen to reflect the distinct risk profiles of the two claim types, $\lambda_1 > \lambda_2$ reflects the higher frequency of type-I claims, while the mean claim sizes $\frac{1}{\alpha_1}=1$ and $\frac{1}{\alpha_2}=5$ reflect the higher severity of type-II claims. The premium rate received by the company is $c = 2.5$. This value satisfies the solvability condition, as the premium rate exceeds the total expected aggregate claim cost per unit time $\frac{\lambda_1}{\alpha_1}+\frac{\lambda_2}{\alpha_2}=2<2.5$. To obtain the numerical results, we substitute these parameters into the general explicit solution derived in Proposition 3. Specifically, the ruin probability functions presented below are calculated using Eq. (40), with the coefficients $\varpi_{k,j}$ determined by Eq. (41) and the exponents $-R_j$ derived from the roots of the Lundberg's equation. The analytical expression for the ruin probability, $\psi(u)$, will be calculated by computing the ruin probabilities caused by claim type-I and claim type-II, and then summing the results (derived with python). This is done for the following copula parameter scenarios. The resulting specific formulas for each dependence scenario are as follows:

a. Independent case ($\theta_1=\theta_2=0$)

$$\psi^{ind}(u) = 0.0792885558e^{-0.65934664u} + 0.720685677e^{-0.06067362u}. \quad (48)$$

b. Positive dependence case ($\theta_1 = 0.5$ and $\theta_2 = 0.5$)

$$\psi^{pos}(u) = 0.0116759934e^{-1.89367845u} + 0.0877710888e^{-0.73066379u} + 0.040477877e^{-0.35581965u} + 0.564344106e^{-0.09610876u}. \quad (49)$$

c. Negative dependence case ($\theta_1 = -0.5$ and $\theta_2 = -0.5$)

$$\psi^{pos}(u) = -0.00353898483e^{-2.09502807u} + 0.0757249995e^{-0.58228366u} - 0.0330136003e^{-0.45446869u} + 0.847300244e^{-0.03276363u}. \quad (50)$$

d. Hybrid dependence I case ($\theta_1 = 0.5$ and $\theta_2 = -0.5$)

$$\psi^{hybI}(u) = 0.00573757401e^{-1.89322719u} + 0.055817724e^{-0.70970477u} - 0.0171933897e^{-0.43949277u} + 0.81192188e^{-0.03893979u}. \quad (51)$$

e. Hybrid dependence II case ($\theta_1 = -0.5$ and $\theta_2 = 0.5$)

$$\psi^{hybII}(u) = -0.0077826902e^{-2.09472779u} + 0.090208067e^{-0.62769491u} + 0.0468799411e^{-0.34837961u} + 0.618983132e^{-0.08792132u}. \quad (52)$$

The resulting specific formulas for the total ruin probability $\psi(u)$ for each dependence scenario are presented in Eqs. (48)-(52). It is important to note that each of these total probability functions is the sum of the individual ruin probabilities caused by type-I and type-II claims $\left(\psi(u) = \psi_1(u) + \psi_2(u)\right)$. While we present the aggregate closed form solutions here for brevity, the behaviors of the individual components $\psi_1(u)$ and $\psi_2(u)$ are analyzed and visualized in Figure 2**.**

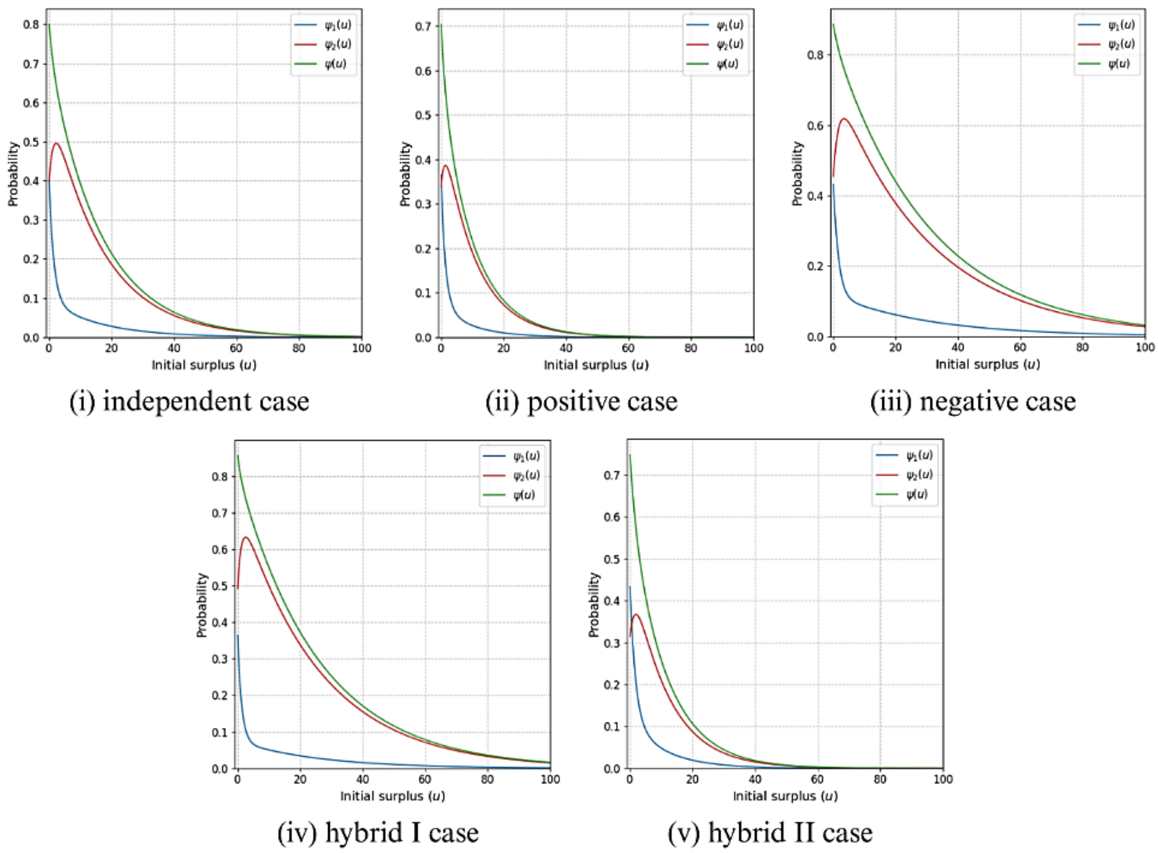


Figure 2. A comparative analysis of ruin probabilities across five dependence scenarios.

Figure 2 provides a visual decomposition of the analytical results presented in Eqs. (48)-(52). While the equations above quantify the total solvency risk, this figure separates the contribution of each claim type. Specifically, the hump observed in the $\psi_2(u)$ curve corresponds to the specific dominant exponential terms in the explicit formula for type-II claims, which differ from the rapidly decaying terms governing type-I claims. Consistently across all subplots, the curve for $\psi_1(u)$ decreases sharply, whereas $\psi_2(u)$ exhibits a characteristic hump and decays much more slowly. This distinct behavior is driven by the fundamental difference in severity. type-I claims, being small, are quickly absorbed by the premium income as the initial surplus u increases. In contrast, type-II claims are high-severity events. Even with a moderate initial surplus, the risk of a single catastrophic claim wiping out the capital remains significant, causing $\psi_2(u)$ to contribute more heavily to ruin at intermediate and high surplus levels. For independent case, The results obtained in this scenario are validated against the results found in Example 3.1 by Han et al. [18]. Figure 3 illustrates the significant impact of different dependence structures on the ruin probability. The negative dependence scenario consistently yields the highest ruin probability, representing the most perilous risk profile. Intuitively, this occurs because negative correlation pairs large claim amounts with short inter-arrival times. Consequently, the insurer faces significant capital outflows before sufficient premium income has been accumulated to absorb the shock, drastically increasing the likelihood of insolvency. Conversely, the positive dependence scenario produces the lowest risk, implying the most favorable condition for solvency. This safety arises because large claims are associated with longer inter-arrival times. This delay provides the insurer with a crucial recovery period to build up premium reserves. This accumulated buffer acts as a financial cushion, making the surplus more resilient when a large loss eventually occurs. The independent case serves as an essential benchmark, situated between these extremes, highlighting that ignoring correlation structures can lead to a significant misestimation of risk..

Table 1. Ruin probability with varying dependence scenario at $u = 10$

Scenario	$\psi_1(u)$	$\psi_2(u)$	$\psi(u)$
Independent	0.05117228	0.34184081	0.39301309
Positive dependence	0.02621920	0.19084138	0.21706058
Negative dependence	0.08533234	0.52512682	0.61045916
Hybrid dependence I	0.05056335	0.49931905	0.54988240
Hybrid dependence II	0.04832009	0.21023392	0.25855400

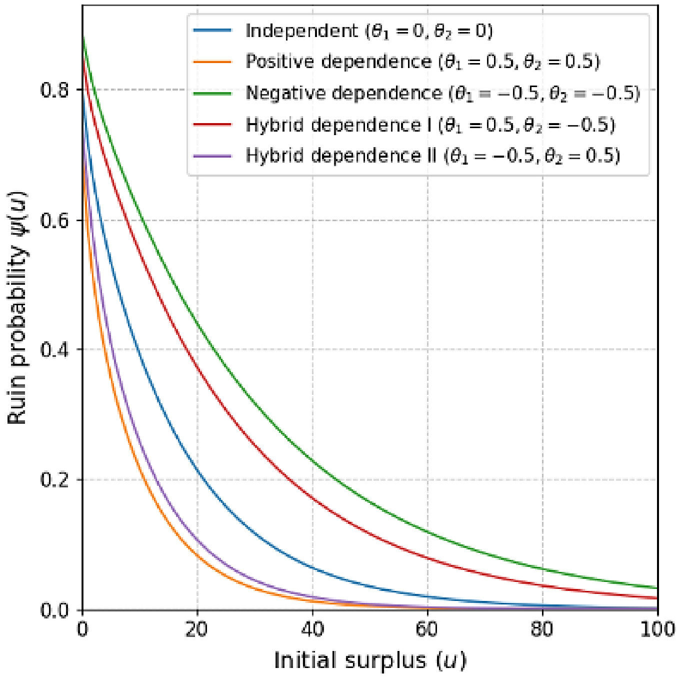


Figure 3. The effect of different dependence structures on the ruin probability, $\psi(u)$, for a given initial surplus, u.

To substantiate the visual findings with concrete numerical evidence, Table 1 presents the ruin probability at a specific surplus point, namely $u = 10$. The data reveals a substantial divergence in risk levels, the ruin probability in the negative dependence case (0.610) is nearly three times higher than in the positive dependence case (0.217). This magnitude underscores that ignoring dependence structures can lead to severe capital underestimation. Furthermore, the hybrid scenarios provide a crucial insight into which claim type drives the overall risk. Comparing the two hybrid cases, hybrid I ($\theta_1 > 0, \theta_2 < 0$) yields a much higher ruin probability (0.550) than hybrid II ($\theta_1 < 0, \theta\_2 > 0$) (0.259). This stark difference indicates that the dependence structure of type-II claims acts as the dominant factor. Specifically, when type-II claims have a negative dependence (as in hybrid I), the risk escalates drastically, regardless of the behavior of type-I claims. Practically, this implies that insurers must prioritize modeling the dependence of high-severity claim lines, as misjudging this specific correlation poses the greatest threat to solvency.

The findings from Example 1 align with established literature [8-12], confirming that negative dependence poses the greatest threat to solvency while positive dependence offers a protective effect. However, this study extends those insights by explicitly isolating the mechanism within a two claim framework. Our analysis demonstrates that the total solvency risk is not equally driven by both claim types. Instead, it is disproportionately dominated by the dependence structure of the high-severity claims. Consequently, a negative dependence within this specific category amplifies the ruin probability far more drastically than similar behavior in type-I claims.

From an actuarial perspective, these results provide critical decision-making directives. The distinct impact of type-II claims implies that insurers cannot rely on aggregate risk models that ignore claim heterogeneity. Instead, capital allocation and pricing strategies must explicitly account for the correlation structure of catastrophic lines. Specifically, if a negative correlation is detected in high-severity claim lines, actuaries must advocate for significantly higher solvency margins or targeted reinsurance coverage to mitigate the elevated risk of rapid capital depletion.

4. CONCLUSIONS

The insurance surplus process is modeled using a continuous-time risk model with two types of claims and FGM copula dependence. In this model, there are two types of claims, named type-I and type-II, with different characteristics in terms of frequency and claim severity. The dependence between inter-claim time and claim size is modeled with an FGM copula. We derive the integro-differential equation for probability ruin. For exponentially distributed claim amounts, an analytic form of the ruin probability is obtained. A numerical illustration confirms that the ruin probability decreases as the initial surplus increases and shows that the FGM copula dependence influences the ruin probability.

A key advantage of this two type risk model is its ability to decompose the solvency risk within a mixed portfolio. By distinguishing between frequent, small claims (type-I) and rare, high-severity claims (type-II), the model reveals that the latter are the primary drivers of ruin, particularly under negative dependence. This distinction is critical for insurers managing mixed portfolios, as traditional aggregate models often obscure the specific correlation risks associated with high-severity lines. By isolating these components, the derived explicit formulas enable actuaries to implement more precise capital allocation and targeted reinsurance strategies. This ensures that reserves are not merely based on average aggregate losses, but are specifically calibrated to withstand the shocks from high-severity components.

This study has two primary limitations that should be considered when interpreting the results. First, the model assumes that the claim arrival for both types follows a Poisson process. While standard, this assumption implies memoryless arrivals, potentially limiting the model's ability to capture risk contagion or seasonal clustering often observed in real world data. Second, the use of the FGM copula limits the analysis to weak dependence structures between inter-claim times and claim sizes. Consequently, the derived ruin probabilities may underestimate the true solvency risk in scenarios characterized by strong tail dependence, such as during major catastrophic events. Based on these limitations, future research could be extended in several specific directions. First, to overcome the weak dependence constraint of the FGM copula, future studies should employ Archimedean copulas such as Clayton, Gumbel, or Frank copula. These copula families allow for the modeling of stronger tail dependence, which is critical for capturing the correlation between extreme events. Second, the assumption of exponentially distributed claim sizes could be relaxed to better fit real world data. Extending the model to incorporate heavy-tailed distributions (e.g., Pareto or Weibull), particularly for type-II claims, would provide a more realistic assessment of solvency risk. Finally, the claim arrival assumption could be generalized by replacing the Poisson process with a renewal process (e.g., Erlang or Cox processes) or a Hawkes process to capture potential clustering and risk contagion effects.

5. ACKNOWLEDGMENTS

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